

Hamiltonian dynamics and Faddeev-Jackiw quantization of 3D gravity with a Barbero-Immirzi like parameter

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A detailed Dirac's and Faddeev-Jackiw quantization of Bonzom-Livine model describing gravity in three dimensions is performed. The full structure of the constraints, the gauge transformations and the generalized Faddeev-Jackiw brackets are found. In addition, we show that the Faddeev-Jackiw and Dirac's brackets coincide to each other. Finally we discuss some remarks and prospects.

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I. INTRODUCTION

It is well-known that 3-dimensional gravity is an interesting toy model. In fact, it is considered as good test theory for trying to understand the difficulties that emerge in the quantization of four dimensional gravity. It is worth to mention, that since the works developed by Achúcarro, Towsend, Witten and other authors [1, 2], there is a huge effort for understanding at classical and quantum level the connection between gravity and gauge connections theories just like Chern-Simons theory [3], then, it is expected that the learned in the three dimensional case could be useful for constructing better tools and apply it for the quantization of four-dimensional gravitational theory. In this respect, it is common to obtain in three dimensions a relation between Palatini and Chern-Simons theory, in fact, it has been showed that these theories are equivalent up to a total derivative [2–4]. However, the relation reported between these theories is not the only one, there is a new action classically equivalent to Palatini's theory, it is the so-called exotic action with a Barbero-Immirzi like parameter [5] (we call it from now on Bonzom-Livine action [BL]). In fact, [BL] model describes a set of actions sharing the same equations of motion with Palatini's theory, however, the symplectic structure is different to each other. The symplectic structure in [BL] model depends of a Barbero-Immirzi like parameter, from which, one expects that the quantum

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theories will be different [4]. In this respect, something similar happens in the four dimensional case with the Holst action [6]. The Holst action provides a set of actions classically equivalent to Einstein's theory, it depends of a parameter called Barbero-Immirzi (we call it γ parameter) and the contribution of this parameter can be appreciated at classical level in the symplectic structure of the theory and the coupling of fermionic matter with gravity, in fact, it determines the coupling constant of a four-fermion interaction [7]. From the quantum point of view, the γ parameter gives a contribution in the quantum spectra of the area and volume operators in the Loop Quantum Gravity context [8]. Furthermore, the term added by Holst to Palatini's action facilitates so much the canonical description of General Relativity, and depending of the values of γ we can reproduce the different scenarios found in canonical gravity, for instance, it is possible to obtain the ADM, Ashtekar and Barbero formulations in straightforward way [6]. Nevertheless, in spite of the Holst action provide a general action for gravity, the γ parameter is still controversial [8]. In this manner, the [BL] action becomes to be the three dimensional equivalent model to Holst's action, in fact, the equivalence is not given only with the presence of an γ parameter, but also at classical level if we perform a partial gauge fixing in the canonical description of [BL], then it is possible to obtain a full Ashtekar's connection dynamics in three dimensions [4]. In this respect, the analysis of the symmetries of the [BL] action has been performed in [4, 5], in these works the canonical analysis by using the Dirac method was performed. However, in these works the analysis was developed on a smaller phase space and the complete structure of the constraints on the full phase space was not reported. It is important to remark that if some of the Dirac steps is omitted, then it is possible to obtain incomplete results [9, 10]. In this manner, an analysis developed on the full phase space and following all the Dirac steps is mandatory. However, in some cases, to develop the Dirac method for gauge theories is large and tedious task, hence, because of these complications, it is necessary to use alternative formulations that could give us a complete canonical description of the theory, in this sense, there is a different approach for studying gauge theories, it is called the Faddeev-Jackiw [FJ] formalism [11]. The [FJ] method is a symplectic approach, namely, all the relevant information of the theory can be obtained through an invertible symplectic tensor, which is constructed by means the symplectic variables that are identified as the degrees of freedom. Because of the theory under interest is singular there will be constraints, and [FJ] has the advantage that all the constraints of the theory are at the same footing, namely, it is not necessary to perform the classification of the constraints in primary, secondary, first class or second class as in Dirac's method is done [12]. When the symplectic tensor is obtained, then its components are identified with the [FJ] generalized brackets, Dirac's brackets and [FJ] brackets coincide to each other.

Because of the explained above, in this paper we develop a pure Dirac's method and a full [FJ] analysis of the [BL] model. In fact, in order to compare both approaches, it is necessary to work in Dirac's method with the complete configuration space. Hence, for constructing the Dirac brackets and compare it with the generalized [FJ] ones, we need to know the complete structure of the constraints over the full phase space [13]. Furthermore, we shall prove that the [FJ] approach is more economic than Dirac's one. It is important to comment that our results has not been reported

in the literature and as special case we reproduce those reported in [4, 5]. In addition, we would also remark that for [BL] theory we shall construct the Dirac brackets by eliminating the second class constraints and remaining the first class ones. Furthermore, at the end of the paper, we have added an appendix where the analysis of an Abelian [BL] theory is performed, in that appendix, we construct the Dirac brackets by fixing the gauge and also we reproduce all those results by means of [FJ] formalism.

The paper is organized as follows; in Section II a detailed canonical analysis of [BL] is performed. We report the complete structure of the constraints defined on the full phase space, then we eliminate the second class constraints by constructing the Dirac brackets. In Section III, we study the relation between [BL] and Chern-Simons theory. We reproduce the results of the previous section by performing a pure Dirac's analysis of a generalized Chern-Simons theory. In the Section IV, a detailed [FJ] of [BL] action is developed. In order to reproduce all the Dirac results, we work with the configuration space field as symplectic variables, we identify all the constraints of the theory and we show that the [FJ] generalized and Dirac's brackets coincide each to other. In Section V we add some remarks and conclusions.

II. HAMILTONIAN DYNAMICS FOR THREE DIMENSIONAL BL GRAVITY

In this section, we will study the Hamiltonian dynamics of the action proposed by [BL] [5]. We will perform our analysis by using a pure Dirac's method, namely, we will find all the constraints defined on the full phase space. As was comment above, there is an analysis of the [BL] action developed on a smaller phase space reported in [4, 5], however, in those works the structure of the constraints is not complete, thus, in order to compare the [FJ] method with the Dirac one it is mandatory to perform the Dirac analysis on the full phase space by following all the Dirac steps.

It is well-known that three dimensional gravity with a cosmological constant can be written as a Chern-Simons theory [1–5]. In fact, if the principal gauge bundle G over M is given by $G = SU(2)$ for 3d Euclidean gravity, then we can enlarge the group G to \tilde{G} , where \tilde{G} could be $SO(4)$, $ISO(3)$ or $SO(3, 1)$ depending on the sign of the cosmological constant Λ positive, zero or negative respectively. Hence, the algebra of the generators of \tilde{G} will satisfy the following commutation relations [4, 5]

$$[J_i, J_j] = \epsilon_{ij}^{k} J_k \quad [J_i, K_j] = \epsilon_{ij}^{k} K_k \quad [K_i, K_j] = s\epsilon_{ij}^{k} J_k, \quad (1)$$

where $s = -1, 0, 1$, corresponding to the sign of the cosmological constant and $i, j, k = 1, 2, 3$. In order to construct a Chern-Simons theory being equivalent to standard Einstein's action of gravity, we choose the following non-degenerate invariant bilinear form

$$\langle J_i, K_j \rangle = \delta_{ij}, \quad \langle J_i, J_j \rangle = \langle K_i, K_j \rangle = 0, \quad (2)$$

in this manner, 3d Palatini's action with cosmological constant can be written as

$$S'_{Palatini} = \frac{1}{\sqrt{|\Lambda|}} \int_M \epsilon^{\mu\nu\rho} \left(\langle A_\mu, \partial_\nu A_\rho \rangle + \frac{1}{3} \langle A_\mu, [A_\nu, A_\rho] \rangle \right). \quad (3)$$

On the other hand, if $\Lambda \neq 0$, then there is another invariant non-degenerate bilinear form given by

$$(J_i, J_j) = \delta_{ij} \quad (K_i, K_j) = s\delta_{ij} \quad (J_i, K_j) = 0, \quad (4)$$

in this case, we can obtain from the following Chern-Simons action

$$\tilde{S}_{Exotic} = \frac{1}{\sqrt{|\Lambda|}} \int_M \epsilon^{\mu\nu\rho} \left((A_\mu, \partial_\nu A_\rho) + \frac{1}{3} (A_\mu, [A_\nu, A_\rho]) \right), \quad (5)$$

the so-called exotic action for gravity

$$\tilde{S}[A, e] = \frac{1}{\sqrt{|\Lambda|}} \left[\int_M A^i \wedge dA_i + \frac{1}{3} \epsilon_{ijk} A^i \wedge A^j \wedge A^k \right] + s\sqrt{|\Lambda|} \int_M e^i \wedge d_A e_i, \quad (6)$$

where the 1-form $A^i = A_\mu^i dx^\mu$, $(d_A v)^i = dv^i + [A, v]^i = dv^i + \epsilon^i_{jk} A^j \wedge v^k$ and $F^i = dA^i + \frac{1}{2} \epsilon^i_{jk} A^j \wedge A^k$ is the strength two-form. In this manner, the [BL] model consists of considering the combination of Palatini's and exotic action through a parameter, namely γ , being a kind of Barbero-Immirzi parameter

$$S_\gamma[A, e] = S'_{Palatini}[A, e] + \frac{1}{\gamma} \tilde{S}_{Exotic}[A, e]. \quad (7)$$

In fact, from the action (7) we obtain a family of theories classically equivalent to 3d gravity in the sense that Palatini's theory with cosmological constant and [BL] actions share the same equations of motion, which can be seen from the variation of the action (7)

$$\frac{\delta S_\gamma[A, e]}{\delta e_{\mu i}} : \quad \epsilon^{\mu\nu\rho} \left[F^i{}_{\nu\rho}[A] + s \frac{|\Lambda|}{2} \epsilon^i{}_{jk} e_\nu^j e_\rho^k \right] + s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{\mu\nu\rho} D_\nu e^i{}_\rho = 0, \quad (8)$$

$$\frac{\delta S_\gamma[A, e]}{\delta A_{\mu i}} : \quad \epsilon^{\mu\nu\rho} D_\nu e^i{}_\rho + \frac{1}{\gamma \sqrt{|\Lambda|}} \epsilon^{\mu\nu\rho} \left[F^i{}_{\nu\rho}[A] + s \frac{|\Lambda|}{2} \epsilon^i{}_{jk} e_\nu^j e_\rho^k \right] = 0, \quad (9)$$

the equations (8) and (9) are equivalent to Einstein's equations. Hence, in order to develop the Hamiltonian analysis, we perform the 2+1 decomposition of the action (7) obtaining

$$\begin{aligned} S_\gamma[e, A] = & \int d^3x \left[2\epsilon^{0ab} \delta_{ij} (e_0^i + \frac{1}{\gamma \sqrt{|\Lambda|}} A_0^i) (F^j{}_{ab} + s \frac{|\Lambda|}{2} \epsilon^j{}_{kl} e_a^k e_b^l) + 2\epsilon^{0ab} \delta_{ij} D_a e_b^i (A_0^j + s \frac{\sqrt{\Lambda}}{\gamma} e_0^j) \right. \\ & \left. + 2\epsilon^{0ab} \delta_{ij} (e_b^i \partial_0 A_a^j + \frac{1}{2\gamma \sqrt{|\Lambda|}} A_b^i \partial_0 A_a^j + s \frac{\sqrt{|\Lambda|}}{2\gamma} e_b^i \partial_0 e_a^j) \right], \end{aligned} \quad (10)$$

where $a, b, c = 1, 2$. The definition of the momenta $(\pi^\alpha_i, \Pi^\alpha_i)$ canonically conjugate to (e^i_α, A_α^i) is given by

$$\Pi^\alpha_i = \frac{\delta \mathcal{L}}{\delta \dot{A}_\alpha^i}, \quad \pi^\alpha_i = \frac{\delta \mathcal{L}}{\delta \dot{e}_\alpha^i}. \quad (11)$$

The matrix elements of the Hessian

$$\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu e_\alpha^i) \partial(\partial_\mu e_\beta^i)}, \quad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu e_\alpha^i) \partial(\partial_\mu A_\beta^i)}, \quad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu A_\alpha^i) \partial(\partial_\mu A_\beta^i)}, \quad (12)$$

are identically zero, thus, we expect 18 primary constraints. From the definition of the momenta (11) we identify the following 18 primary constraints

$$\begin{aligned}
\phi_i^0 &:= \pi_i^0 \approx 0, \\
\phi_i^a &:= \pi_i^a - s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} e_b^j \approx 0, \\
\Phi_i^0 &:= \Pi_i^0 \approx 0, \\
\Phi_i^a &:= \Pi_i^a - 2\epsilon^{0ab} \delta_{ij} (e_b^j + \frac{1}{2\gamma\sqrt{|\Lambda|}} A_b^j) \approx 0.
\end{aligned} \tag{13}$$

The canonical Hamiltonian takes the form

$$H_c = \int dx^2 \left[-2\epsilon^{0ab} \delta_{ij} D_a e_b^i (A_0^i + s \frac{\sqrt{\Lambda}}{\gamma} e_0^i) - 2\epsilon^{0ab} \delta_{ij} (e_0^i + \frac{1}{\gamma\sqrt{|\Lambda|}} A_0^i) (F_{ab}^j + s \frac{|\Lambda|}{2} \epsilon_{kl}^j e_a^k e_b^l) \right], \tag{14}$$

and the primary Hamiltonian is given by

$$H_P = H_c + \int dx^2 [\lambda_{\alpha}^i \phi_i^{\alpha} + \xi_{\alpha}^i \Phi_i^{\alpha}], \tag{15}$$

where $\lambda_{\alpha}^i, \xi_{\alpha}^i$ are Lagrange multipliers enforcing the constraints $(\phi_i^{\alpha}, \Phi_i^{\alpha})$. The fundamental Poisson brackets of the theory are given

$$\begin{aligned}
\{e_{\alpha}^i(x), \pi_{\beta}^j(y)\} &= \delta_{\alpha}^{\beta} \delta^i_j \delta^2(x-y), \\
\{A_{\alpha}^i(x), \Pi_{\beta}^j(y)\} &= \delta_{\alpha}^{\beta} \delta^i_j \delta^2(x-y),
\end{aligned} \tag{16}$$

where we can observe that in these fundamental brackets there is not any contribution of the γ parameter; in the Dirac brackets, however, there will be a non trivial contribution. In order to observe the presence of more constraints, we calculate the following 18×18 matrix whose entries are the Poisson brackets among the constraints (13)

$$\begin{aligned}
\{\phi_i^a(x), \phi_j^b(y)\} &= -2s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\
\{\phi_i^a(x), \Phi_j^b(y)\} &= -2\epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\
\{\Phi_i^a(x), \Phi_j^b(y)\} &= -2 \frac{1}{\gamma\sqrt{|\Lambda|}} \epsilon^{0ab} \delta_{ij} \delta^2(x-y),
\end{aligned} \tag{17}$$

we appreciate that this matrix has rank=12 and 6 null-vectors. By using the 6 null-vectors and consistency conditions we obtain the following 6 secondary constraints

$$\begin{aligned}
\gamma_i^0 &= \pi_i^0 \approx 0, \\
\tilde{\gamma}_i^0 &= \Pi_i^0 \approx 0, \\
\dot{\phi}_i^0 &= \{\phi_i^0(x), H_P\} \approx 0 \quad \Rightarrow \quad \psi_i := 2\epsilon^{0ab} s \frac{\sqrt{|\Lambda|}}{\gamma} D_a e_{ib} + 2\epsilon^{0ab} (F_{iab} + \frac{s|\Lambda|}{2} \epsilon_{ijk} e_a^j e_b^k) \approx 0, \\
\dot{\Phi}_i^0 &= \{\Phi_i^0(x), H_P\} \approx 0 \quad \Rightarrow \quad \Psi_i := 2\epsilon^{0ab} D_a e_{ib} + 2\epsilon^{0ab} \frac{1}{\gamma\sqrt{|\Lambda|}} (F_{iab} + \frac{s|\Lambda|}{2} \epsilon_{ijk} e_a^j e_b^k) \approx 0,
\end{aligned} \tag{18}$$

and the rank allows us to fix the following values for the Lagrangian multipliers

$$\begin{aligned}\dot{\phi}_i^a &= \{\phi_i^a, H_P\} \approx 0 \Rightarrow 2\epsilon^{0ab} \frac{s - \gamma^2}{\gamma} \sqrt{|\Lambda|} (-\lambda_{bi} + D_b e_{0i} + \epsilon_{lim} e_b^m A_0^l) \approx 0, \\ \dot{\Phi}_i^a &= \{\Phi_i^a, H_P\} \approx 0 \Rightarrow 2\epsilon^{0ab} \frac{s - \gamma^2}{\gamma^2} (-\xi_{bi} + D_b A_{0i} + s |\Lambda| \epsilon_{lim} e_b^m e_0^l) \approx 0.\end{aligned}\quad (19)$$

Consistency requires conservation in time of the secondary constraints, however, for this theory there are not third constraints. At this point, we need to identify from the primary and secondary constraints which ones correspond to first and second class. For this aim, we need to calculate the rank and the null-vectors of the 24×24 matrix whose entries will be the Poisson brackets between primary and secondary constraints, this is

$$\begin{aligned}\{\phi_i^a(x), \phi_j^b(y)\} &= -2s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{\phi_i^a(x), \Phi_j(y)\} &= -2\epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{\phi_i^a(x), \psi_j(y)\} &= -2\epsilon^{0ab} \left[\frac{s\sqrt{|\Lambda|}}{\gamma} \delta_{ij} \partial_b - \epsilon_{ijk} \left(\frac{s\sqrt{|\Lambda|}}{\gamma} A_b^k + s |\Lambda| e_b^k \right) \right] \delta^2(x - y), \\ \{\phi_i^a(x), \Psi_j(y)\} &= -2\epsilon^{0ab} \left[\delta_{ij} \partial_b - \epsilon_{ijk} \left(A_b^k + \frac{s\sqrt{|\Lambda|}}{\gamma} e_b^k \right) \right] \delta^2(x - y), \\ \{\Phi_i^a(x), \Phi_j^b(y)\} &= -2 \frac{1}{\gamma \sqrt{|\Lambda|}} \epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{\Phi_i^a(x), \psi_j(y)\} &= -2\epsilon^{0ab} \left[\delta_{ij} \partial_b - \epsilon_{ijk} \left(A_b^k + \frac{s\sqrt{|\Lambda|}}{\gamma} e_b^k \right) \right] \delta^2(x - y), \\ \{\Phi_i^a(x), \Psi_j(y)\} &= -2\epsilon^{0ab} \left[\frac{1}{\gamma \sqrt{|\Lambda|}} \delta_{ij} \partial_b - \epsilon_{ijk} \left(\frac{1}{\gamma \sqrt{|\Lambda|}} A_b^k + e_b^k \right) \right] \delta^2(x - y), \\ \{\psi_i(x), \psi_j(y)\} &= 0, \\ \{\Psi_i(x), \Psi_j(y)\} &= 0, \\ \{\psi_i(x), \Psi_j(y)\} &= 0,\end{aligned}\quad (20)$$

This matrix has a rank=12 and 12 null vectors, thus, the theory presents a set of 12 first class constraints and 12 second class constraints. By using the contraction of the null vectors with the constraints (13) and (18), we identify the following 12 first class constraints

$$\begin{aligned}\gamma_i^0 &= \pi_i^0 \approx 0, \\ \Gamma_i^0 &= \Pi_i^0 \approx 0, \\ \omega_i &= D_a \chi_i^a - s |\Lambda| \epsilon_i^j e_a^k \Xi_j^a + 2\epsilon^{0ab} s \frac{\sqrt{|\Lambda|}}{\gamma} D_a e_{ib} + 2\epsilon^{0ab} (F_{iab} + \frac{s |\Lambda|}{2} \epsilon_{ijk} e_a^j e_b^k) \approx 0, \\ \Gamma_i &= D_a \Xi_i^a - \epsilon_i^j e_a^k \chi_j^a + 2\epsilon^{0ab} D_a e_{ib} + 2\epsilon^{0ab} \frac{1}{\gamma \sqrt{|\Lambda|}} (F_{iab} + \frac{s |\Lambda|}{2} \epsilon_{ijk} e_a^j e_b^k) \approx 0,\end{aligned}\quad (21)$$

and the following 12 second class constraints

$$\begin{aligned}\chi_i^a &= \pi_i^a - s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} e_b^j \approx 0, \\ \Xi_i^a &= \Pi_i^a - 2\epsilon^{0ab} \delta_{ij} (e_b^j + \frac{1}{2\gamma \sqrt{|\Lambda|}} A_b^j) \approx 0.\end{aligned}\quad (22)$$

It is important to remark that these constraints have not been reported in the literature, and their complete structure defined on the full phase space will be relevant in order to know the fundamental gauge transformations and for constructing the Dirac brackets. On the other hand, the constraints will play a key role to make the progress in the quantization. We have commented above that the structure of the constraints (21) is obtained by means of the null vectors, for instance, a set of null vectors of the matrix (20) are given by

$$V_i^1 = (0, -\delta_j^i \partial_a \delta^2(x-y) - \epsilon_{jk}^i A_a^k \delta^2(x-y), 0, -s \mid \Lambda \mid \epsilon_i^j \epsilon_k^a \delta^2(x-y), \delta_j^i \delta^2(x-y), 0),$$

hence, by performing the contraction of these null vectors with the primary and secondary constraints, the first class constraint ω_i given in (21) is obtained.

Now, we will calculate the algebra of the constraints

$$\begin{aligned} \{\chi_i^a(x), \chi_j^b(y)\} &= -2s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{\chi_i^a(x), \Xi_j^b(y)\} &= -2\epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{\Xi_i^a(x), \Xi_j^b(y)\} &= -2 \frac{1}{\gamma \sqrt{|\Lambda|}} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\ \{\chi_i^a(x), \omega_j(y)\} &= s \mid \Lambda \mid \epsilon_{ij}^k \Phi_k^a \delta^2(x-y) \approx 0, \\ \{\Xi_i^a(x), \omega_j(y)\} &= \epsilon_{ij}^k \phi_k^a \delta^2(x-y) \approx 0, \\ \{\chi_i^a(x), \Gamma_j(y)\} &= \epsilon_{ij} \phi_k^a \delta^2(x-y) \approx 0, \\ \{\Xi_i^a(x), \Gamma_j(y)\} &= \epsilon_{ijk} \Phi^{ka} \delta^2(x-y) \approx 0, \\ \{\omega_i(x), \omega_j(y)\} &= s \mid \Lambda \mid \epsilon_{ijk} \Gamma^k \delta^2(x-y) \approx 0, \\ \{\Gamma_i(x), \Gamma_j(y)\} &= \epsilon_{ijk} \Gamma^k \delta^2(x-y) \approx 0, \\ \{\omega_i(x), \Gamma_j(y)\} &= \epsilon_{ijk} \omega^k \delta^2(x-y) \approx 0. \end{aligned} \tag{23}$$

where we can observe that the algebra is closed. Furthermore, with all the information obtained until now, we can construct the Dirac brackets. In fact, there are two options for constructing them, the first way is by eliminating the second class constraints and keeping on the first class, the second option is by fixing the gauge and converting the first class constraints into second class ones. In this section we will eliminate the second class constraints remaining the first class ones; in the [FJ] approach we will discuss both. Hence, in order to construct the Dirac brackets, we shall use the matrix whose elements are only the Poisson brackets among second class constraints, namely $C_{\alpha\beta}(u, v) = \{\zeta^\alpha(u), \zeta^\beta(v)\}$, given by

$$[C_{(\alpha\beta)}(x, x')]^{ab}_{ij} = -2 \begin{pmatrix} \frac{s\sqrt{|\Lambda|}}{\gamma} & 1 \\ 1 & \frac{1}{\gamma\sqrt{|\Lambda|}} \end{pmatrix} \delta_{ij} \epsilon^{0ab} \delta^2(x-x'), \tag{24}$$

its inverse is given by

$$[C^{-1}_{(\alpha\beta)}(x, x')] = \frac{\gamma^2}{2(s-\gamma^2)} \begin{pmatrix} \frac{1}{\gamma\sqrt{|\Lambda|}} & -1 \\ -1 & \frac{s\sqrt{|\Lambda|}}{\gamma} \end{pmatrix} \delta^{ij} \epsilon_{0ba} \delta^2(x-x'). \tag{25}$$

The Dirac brackets among two functionals A, B are expressed by

$$\{A(x), B(y)\}_D = \{A(x), B(y)\}_P - \int dudv \{A(x), \zeta^\alpha(u)\} C^{-1}_{\alpha\beta}(u, v) \{\zeta^\beta(v), B(y)\}, \quad (26)$$

where $\{A(x), B(y)\}_P$ is the usual Poisson bracket between the functionals A, B and $\zeta^\alpha(u) = (\chi_i^a, \Xi_i^a)$ is the set of second class constraints. Hence, by using (25) and (26) we obtain the following Dirac's brackets of the theory

$$\begin{aligned} \{e^i_a(x), \pi^b_j(y)\}_D &= \frac{s}{2(s-\gamma^2)} \delta^b_a \delta^i_j \delta^2(x-y), \\ \{e^i_a(x), e^j_b(y)\}_D &= \frac{1}{2\sqrt{|\Lambda|}} \frac{\gamma}{s-\gamma^2} \delta^{ij} \epsilon_{0ab} \delta^2(x-y), \\ \{\pi^a_i(x), \pi^b_j(y)\}_D &= \frac{s^2 \sqrt{|\Lambda|}}{2\gamma s-\gamma^2} \delta_{ij} \epsilon^{0ab} \delta^2(x-y), \\ \{A^i_a(x), \Pi^b_j(y)\}_D &= \frac{s-2\gamma^2}{2(s-\gamma^2)} \delta^b_a \delta^i_j \delta^2(x-y), \\ \{A^i_a(x), A^j_b(y)\}_D &= \frac{s\sqrt{|\Lambda|}}{2} \frac{\gamma}{s-\gamma^2} \delta^{ij} \epsilon_{0ab} \delta^2(x-y), \\ \{\Pi^a_i(x), \Pi^b_j(y)\}_D &= \frac{s}{2\gamma\sqrt{|\Lambda|}} \frac{1}{s-\gamma^2} \delta_{ij} \epsilon^{0ab} \delta^2(x-y), \\ \{A^i_a(x), e^j_b(y)\}_D &= \frac{1}{2} \frac{\gamma^2}{\gamma^2-s} \delta^{ij} \epsilon_{0ab} \delta^2(x-y), \\ \{e^i_a(x), \Pi^b_j(y)\}_D &= \frac{1}{2\sqrt{|\Lambda|}} \frac{\gamma}{s-\gamma^2} \delta^b_a \delta^i_j \delta^2(x-y), \\ \{A^i_a(x), \pi^b_j(y)\}_D &= -\frac{s\sqrt{|\Lambda|}}{2} \frac{\gamma}{s-\gamma^2} \delta^b_a \delta^i_j \delta^2(x-y), \\ \{\pi^a_i(x), \Pi^b_j(y)\}_D &= \frac{1}{2} \frac{s}{s-\gamma^2} \delta_{ij} \epsilon^{0ab} \delta^2(x-y). \end{aligned} \quad (27)$$

We can observe that the Dirac brackets depend of the constants (s, γ) and we can reproduce several scenarios depending of the values of these constants. In fact, if we take $s = 1$ and the limit $\gamma \rightarrow \infty$ we recover the Dirac canonical structure of Palatini's action, for instance, that reported in [9]. It is important to remark that in [BL] model the fields e, A and its canonical momenta have become non-commutative while in Palatini's action they are commutative, this is a classical difference between [BL] and Palatini's theory. Moreover, at quantum level this difference is fundamental for constructing the Ashtekar representation of [BL] model [4].

Now, we can calculate the gauge transformations on the full phase space. In fact, the correct gauge symmetry is obtained according to Dirac's conjecture by constructing a gauge generator using the first class constraints, and the structure of the constraints defined on the full phase space will give us the fundamental gauge transformations. For this aim, we will apply the Castellani's algorithm to construct the gauge generator; we define the generator of gauge transformations as

$$G = \int_{\Sigma} [D_0 \varepsilon^i{}_0 \gamma^0{}_i + D_0 \tau_0^i \Gamma^0{}_i + \varepsilon^i \omega_i + \tau^i \Gamma_i]. \quad (28)$$

Therefore, we find that the gauge transformations on the phase space are

$$\begin{aligned}
\delta_0 e^i{}_0 &= D_0 \varepsilon^i{}_0, \\
\delta_0 e^i{}_a &= -D_a \varepsilon^i + \epsilon^i{}_{jk} e^k{}_a \tau^j, \\
\delta_0 A_0^i &= D_0 \tau_0^i, \\
\delta_0 A_a^i &= -D_a \tau^i + s \mid \Lambda \mid \epsilon^i{}_{jk} e^k{}_a \varepsilon^j, \\
\delta_0 \pi^0{}_i &= 0, \\
\delta_0 \pi^a{}_i &= -\Omega \epsilon^{0ab} D_b \varepsilon_i + \epsilon_{ij}{}^k \pi^a{}_k \tau^j + s \mid \Lambda \mid \epsilon_{ij}{}^k \Xi^a{}_k, \\
\delta_0 \Pi^0{}_i &= -\epsilon_{ij}{}^k (\pi^0{}_k \varepsilon^j - \Pi^0{}_k \tau^j), \\
\delta_0 \Pi^a{}_i &= -2\epsilon^{0ab} D_b \varepsilon_i + \epsilon_{ij}{}^k \chi^a{}_k \varepsilon^j - \frac{1}{\gamma \sqrt{\mid \Lambda \mid}} \epsilon^{0ab} D_b \tau_i + \epsilon_{ij}{}^k \Xi^a{}_k \tau^j \\
&\quad + 2\epsilon^{0ab} \epsilon_{ijk} e^k{}_b \tau^j + \Omega \epsilon^{0ab} \epsilon_{ijk} e^k{}_b \varepsilon^j.
\end{aligned} \tag{29}$$

We realize that the fundamental gauge transformations of the [BL] action are given by (29) and they are an Λ -deformed $ISO(3)$ transformations. It is important to remark, that the internal group of the theory is $SU(2)$ (or $SO(3)$), however, the fundamental gauge symmetry and they correspond to Λ -deformed $ISO(3)$ transformations, all these results were not reported in [4, 5]. On the other hand, any theory with background independence is diffeomorphisms covariant, and this symmetry must be obtained from the fundamental gauge transformations. Hence, the diffeomorphisms can be found by redefining the gauge parameters as $\varepsilon_0^i = -\varepsilon^i = \xi^\rho e^i{}_\rho$, $\tau_0^i = -\tau^i = \xi^\rho A_\rho^i$, and the gauge transformation (29) takes the following form

$$\begin{aligned}
e'^i{}_\alpha &\rightarrow e^i{}_\alpha + \mathfrak{L}_\xi e^i{}_\alpha + \xi^\rho [D_\alpha e^i{}_\rho - D_\rho e^i{}_\alpha], \\
A'^i{}_\alpha &\rightarrow A_\alpha^i + \mathfrak{L}_\xi A_\alpha^i + \xi^\rho [\partial_\alpha A_\rho^i - \partial_\rho A_\alpha^i + \epsilon_{ijk} A^j{}_\alpha A^k{}_\rho + s \mid \Lambda \mid \epsilon^i{}_{jk} e^j{}_\alpha e^k{}_\rho],
\end{aligned} \tag{30}$$

Therefore, diffeomorphisms are obtained (on shell) from the fundamental gauge transformations as an internal symmetry of the theory. With the correct identification of the constraints, we can carry out the counting of degrees of freedom in the following form: there are 36 canonical variables $(e^i{}_\alpha, A_\alpha^i, \pi^\alpha{}_i, \Pi^\alpha{}_i)$, 12 first class constraints $(\gamma_i^0, \Gamma_i^0, \omega_i, \Gamma_i)$ and 12 second class constraints (χ_i^a, Ξ_i^a) and one concludes that the $S_\gamma[A, e]$ action for gravity in three dimensions is devoid of degrees of freedom, therefore, the theory is topological.

As a conclusion of this part, we have obtained the extended action, the extended Hamiltonian, the complete structure of the constraints on the full phase space, and the algebra among them. The price to pay for working on the complete phase space, is that the theory presents a set of first and second class constraints; by using the second class constraints we have constructed the Dirac brackets and they will be useful in the quantization of the theory [4].

III. RELATION WITH CHERN-SIMONS THEORY

The action analyzed in the previous section, can be written in an elegant form in terms of a Chern-Simons theory. In fact, by introducing the following variables $\omega^{\pm i} = A^i \pm \sigma \sqrt{|\Lambda|} e^i$ [5], where $\sigma^2 = s$ with $\sigma = 1$ for $\Lambda > 0$ and $\sigma = i$ for $\Lambda < 0$, we obtain that the action (7) can be written as

$$S_\gamma = \frac{\sqrt{|\Lambda|}}{2} \left(\gamma^{-1} + \frac{\sigma}{s} \right) S_{CS}(\omega^+) + \frac{\sqrt{|\Lambda|}}{2} \left(\gamma^{-1} - \frac{\sigma}{s} \right) S_{CS}(\omega^-), \quad (31)$$

where

$$S_{CS}(\omega^\pm) = \int_M \omega^{\pm i} \wedge d\omega_i^\pm + \frac{1}{3} \epsilon_{ijk} \omega^{\pm i} \wedge \omega^{\pm j} \wedge \omega^{\pm k}.$$

The equations of motion obtained from (31) imply that the connections $\omega^{\pm i}$ are flat, and it is easy to prove that these flatness conditions are equivalent to the equations of motion given in (8)-(9). On the other hand, if we develop a pure Dirac's analysis of the action (31) we will reproduce the results given in the previous section, in particular we will reproduce the Dirac brackets. In fact, in summary by performing a pure Dirac's method we obtain the following results:

There are the following first class constraints

$$\mathfrak{G}_i^\pm = \frac{s \pm \sigma \gamma}{s \gamma \sqrt{|\Lambda|}} \epsilon^{0ab} (\partial_a \omega_{bi}^{(\pm)} + \frac{1}{2} \epsilon_{ijk} \omega_b^{(\pm)j} \omega_c^{(\pm)k}) + D_a^{(\pm)} \chi_i^{(\pm)a} \approx 0, \quad (32)$$

and there are the following second class constraints

$$\chi_i^{(\pm)a} = \pi_i^{(\pm)a} - \frac{1}{2} \frac{s \pm \sigma \gamma}{s \gamma \sqrt{|\Lambda|}} \epsilon^{0ab} \omega_{(\pm)bi} \approx 0, \quad (33)$$

here, $\pi_i^{(\pm)a}$ is the conjugate canonical momenta of the connection $\omega_a^{\pm i}$ and $D_a^\pm \lambda^i = \partial_a \lambda^i + \epsilon^i_{jk} \omega_a^{\pm j} \lambda^k$. Furthermore, by using the second class constraints (33) the following Dirac's brackets are obtained

$$\begin{aligned} \{\omega_a^{(\pm)i}(x), \omega_b^{(\pm)j}(y)\}_D &= \frac{s \gamma \sqrt{|\Lambda|}}{s \pm \sigma \gamma} \delta^{ij} \epsilon_{0ab} \delta^2(x-y), \\ \{\omega_a^{(\pm)i}(x), \pi_j^{(\pm)b}(y)\}_D &= \frac{1}{2} \delta_j^i \delta_a^b \delta^2(x-y), \\ \{\pi_i^{(\pm)a}(x), \pi_j^{(\pm)b}(y)\}_D &= \frac{1}{4} \frac{s \pm \sigma \gamma}{s \gamma \sqrt{|\Lambda|}} \delta_{ij} \epsilon^{0ab} \delta^2(x-y), \end{aligned} \quad (34)$$

we can observe that the Dirac brackets between dynamical variables given in (34) are depending of the constants (s, γ, σ) , this fact will be important because by using the definition of $\omega_a^{\pm i}$ given in terms of A_a^i and e_a^i into (34), then the Dirac brackets given in (27) are reproduced. It is important to comment that our results are given in a general form and contain the cases for $\Lambda > 0$ and $\Lambda < 0$, thus, in particular we can reproduce the results given in [14] where the case $\Lambda < 0$ was studied. On the other hand, in the limit $\gamma \rightarrow \infty$ the action (31) is reduced to $S_{CS}(\omega)$ and the Dirac brackets are reduced to those reported in [9] where Palatini's theory was analyzed. Finally, we can observe that in this section we have proved the equivalence between [BL] model and the Chern-Simons theory, thus, the standard quantization procedure can be performed. In the following section we will study the action (7) by using the [FJ] approach and we will obtain all the Dirac results in an alternative way.

IV. FADDEEV-JACKIW ANALYSIS FOR BL THEORY

In this section we will develop the [FJ] formalism for the [BL] model, rewriting the action (10) in the following form

$$\mathcal{L} = 2\epsilon^{0ab}\delta_{ij}e_b^i\dot{A}_a^j + \beta\epsilon^{0ab}\delta_{ij}A_b^i\dot{A}_a^j + \Omega\epsilon^{0ab}\delta_{ij}e_b^i\dot{e}_a^j - V^{(0)}, \quad (35)$$

where $V^{(0)} = -2\epsilon^{0ab}\delta_{ij} \left[(e_0^i + \beta A_0^i)(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon_{kl}^j e_a^k e_b^l) + (\Omega e_0^i + A_0^i)D_a e_b^j \right]$ is called the symplectic potential and we have introduced the following constants Ω and β defined by

$$\Omega = \frac{s\sqrt{|\Lambda|}}{\gamma}, \quad \beta = \frac{1}{\gamma\sqrt{|\Lambda|}}. \quad (36)$$

In the [FJ] framework, the Euler-Lagrange equations of motion are given by [11]

$$f_{ab}^{(0)}\dot{\xi}^b = \frac{\partial V^{(0)}(\xi)}{\partial \xi^a}, \quad (37)$$

where the symplectic matrix $f_{ab}^{(0)}$ takes the form

$$f_{ab}^{(0)}(x, y) = \frac{\delta a_b(y)}{\delta \xi^a(x)} - \frac{\delta a_a(x)}{\delta \xi^b(y)}, \quad (38)$$

with $\xi^{(0)a}$ and $a^{(0)}_a$ representing a set of symplectic variables. It is important to comment, that in [FJ] framework we are free to choose the symplectic variables; we can choose the field configuration variables or the phase space variables. In previous sections, we have constructed the Dirac brackets by eliminating the second class constraints, hence, in order to obtain these results by means [FJ] we will use the configuration space as symplectic variables [13]. For this aim, we choose from the symplectic Lagrangian (35) the following symplectic variables $\xi^{(0)a}(x) = \{e_a^i, e_0^i, A_a^i, A_0^i\}$ and the components of the symplectic 1-forms are $a^{(0)}_a(x) = \{\Omega\epsilon^{0ab}e_{bi}, 0, 2\epsilon^{0ab}e_{bi} + \beta\epsilon^{0ab}A_{bi}, 0\}$. Hence, by using our set of symplectic variables, the symplectic matrix (38) takes the form

$$f_{ab}^{(0)}(x, y) = \begin{pmatrix} -2\Omega\epsilon^{0ab}\delta_{ij} & 0 & -2\epsilon^{0ab}\delta_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ -2\epsilon^{0ab}\delta_{ij} & 0 & -2\beta\epsilon^{0ab}\delta_{ij} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x - y). \quad (39)$$

The symplectic matrix $f_{ab}^{(0)}$ is of dimension $[18 \times 18]$ and it is a singular matrix. In fact, in [FJ] method this means that there are present constraints. In order to obtain these constraints, we calculate the zero modes of the symplectic matrix, the modes are given by $(v_a^{(0)})_1^T = (0, v^{e_0^i}, 0, 0)$ and $(v_a^{(0)})_2^T = (0, 0, 0, v^{A_0^i})$, where $v^{e_0^i}$ and $v^{A_0^i}$ are arbitrary functions. In this manner, by using the zero-modes and the symplectic potential $V^{(0)}$ we obtain

$$\begin{aligned} \Omega_i^{(0)} &= \int d^2x (v^{(0)})_a^T(x) \frac{\delta}{\delta \xi^{(0)a}(x)} \int d^2y V^{(0)}(\xi) \\ &= - \int d^2x v^{e_0^i}(x) 2\epsilon^{0ab}\delta_{ij} \left[(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon_{kl}^j e_a^k e_b^l) + \Omega D_a e_b^j \right] \\ &\rightarrow -2\epsilon^{0ab}\delta_{ij} \left[(F_{ab}^j + s\frac{|\Lambda|}{2}\epsilon_{kl}^j e_a^k e_b^l) + \Omega D_a e_b^j \right] = 0, \end{aligned} \quad (40)$$

$$\begin{aligned}
\beta_i^{(0)} &= \int d^2x (v^{(0)})_a^T(x) \frac{\delta}{\delta \xi^{(0)a}(x)} \int d^2y V^{(0)}(\xi) \\
&= - \int d^2x v^{A_0^i}(x) 2\epsilon^{0ab} \delta_{ij} \left[\beta(F^j_{ab} + s \frac{|\Lambda|}{2} \epsilon^j_{kl} e_a^k e_b^l) + D_a e^j_b \right] \\
&\rightarrow -2\epsilon^{0ab} \delta_{ij} \left[\beta(F^j_{ab} + s \frac{|\Lambda|}{2} \epsilon^j_{kl} e_a^k e_b^l) + D_a e^j_b \right] = 0.
\end{aligned} \tag{41}$$

thus we identify the following constraints

$$\Omega_i^{(0)} = 2\epsilon^{0ab} \delta_{ij} \left[(F^j_{ab} + s \frac{|\Lambda|}{2} \epsilon^j_{kl} e_a^k e_b^l) + \Omega D_a e^j_b \right] = 0, \tag{42}$$

$$\beta_i^{(0)} = 2\epsilon^{0ab} \delta_{ij} \left[\beta(F^j_{ab} + s \frac{|\Lambda|}{2} \epsilon^j_{kl} e_a^k e_b^l) + D_a e^j_b \right] = 0, \tag{43}$$

these constraints are the secondary constraints found by means Dirac's method in the above sections.

In order to observe if there are more constraints, we calculate the following [15–18]

$$f_{cb}^{(1)} \dot{\xi}^b = Z_c(\xi), \tag{44}$$

where

$$Z_c(\xi) = \begin{pmatrix} \frac{\partial V^{(0)}(\xi)}{\partial \xi^a} \\ 0 \\ 0 \end{pmatrix}, \tag{45}$$

and

$$f_{cb}^{(1)} = \begin{pmatrix} f_{ab}^{(0)} \\ \frac{\partial \Omega^{(0)}}{\partial \xi^b} \\ \frac{\partial \beta^{(0)}}{\partial \xi^b} \end{pmatrix} = \begin{pmatrix} -2\Omega\epsilon^{0ab}\delta_{ij} & 0 & -2\epsilon^{0ab}\delta_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ -2\epsilon^{0ab}\delta_{ij} & 0 & -2\beta\epsilon^{0ab}\delta_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ 2\epsilon^{0ab}(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(\Omega A_a^k + s|\Lambda|e_a^k)) & 0 & 2\epsilon^{0ab}(\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 \\ 2\epsilon^{0ab}(\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 & 2\epsilon^{0ab}(\beta\delta_{ij}\partial_a - \epsilon_{ijk}(\beta A_a^k + e_a^k)) & 0 \end{pmatrix} \delta^2(x-y). \tag{46}$$

We can observe that the matrix (A9) is not a square matrix as expected, however, it has linearly independent modes given by $(v^{(1)})_1^T = (\delta_j^i \partial_a v^\lambda + \epsilon^i_{kj} A_a^k v^\lambda, \delta_j^i v^{e_0^i}, -s|\Lambda|\epsilon^i_{jk} e_a^k, 0, \delta_j^i v^\lambda, 0)$ and $(v^{(1)})_2^T = (-\epsilon^i_{jk} e_a^k, 0, \delta_j^i \partial_a v^\beta + \epsilon^i_{kj} A_a^k v^\beta, \delta_j^i v^{A_0^i}, 0, \delta_j^i v^\beta)$. These modes are used in order to obtain further constraints. In fact, by calculating the following contraction [15–18]

$$(v^{(1)})_c^T Z_c = 0, \tag{47}$$

where $c = 1, 2$, we obtain that (47) is an identity, thus, in [FJ] formalism there are not more constraints for the theory under study.

Now, we will construct a new symplectic Lagrangian with the information of the constraints obtained in (42) and (43). In order to archive this aim, we introduce $e_0^i = \dot{\lambda}^i$ and $A_0^i = \dot{\theta}^i$, as Lagrange multipliers associated to those constraints, thus, we obtain the following symplectic Lagrangian

$$\mathcal{L}^{(1)} = 2\epsilon^{0ab}\delta_{ij}e_b^i\dot{A}_a^j + \beta\epsilon^{0ab}\delta_{ij}A_b^i\dot{A}_a^j + \Omega\epsilon^{0ab}\delta_{ij}e_b^i\dot{e}_a^j + \Omega_i^{(0)}\dot{\lambda}^i + \beta_i^{(0)}\dot{\theta}^i - V^{(1)}, \quad (48)$$

where $V^{(1)} = V^{(0)}|_{\Omega_i^{(0)}=0, \beta_i^{(0)}=0} = 0$, the symplectic potential vanishes reflecting the general covariance of the theory. In this manner, from (48) we identify the following new symplectic variables $\xi^{(1)a}(x) = \{e_a^i, \lambda^i, A_a^i, \theta^i\}$ and the new symplectic 1-forms $a^{(0)}_a(x) = \{\Omega\epsilon^{0ab}e_{bi}, \Omega_i^{(0)}, 2\epsilon^{0ab}e_{bi} + \beta\epsilon^{0ab}A_{bi}, \beta_i^{(0)}\}$. Hence, by using the new symplectic variables and 1-forms, we can calculate the following symplectic matrix

$$f_{ab}^{(1)}(x, y) =$$

$$\begin{pmatrix} -2\delta_{ij} & -2(\Omega\delta_{ij}\partial_b + \epsilon_{ijk}(\Omega A_b^k + s | \Lambda | e_b^k)) & -2\delta_{ij} & -2(\delta_{ij}\partial_b + \epsilon_{ijk}(A_b^k + \Omega e_b^k)) \\ -2(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(\Omega A_i^k + s | \Lambda | e_a^k)) & 0 & -2(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(A_i^k + \Omega e_a^k)) & 0 \\ -2\delta_{ij} & -2(\delta_{ij}\partial_b + \epsilon_{ijk}(A_b^k + \Omega e_b^k)) & -2\beta\delta_{ij} & -2(\beta\delta_{ij}\partial_b + \epsilon_{ijk}(\beta A_b^k + e_b^k)) \\ -2(\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 & -2(\beta\delta_{ij}\partial_a - \epsilon_{ijk}(\beta A_a^k + e_a^k)) & 0 \end{pmatrix}$$

$$\times \epsilon^{0ab}\delta^2(x - y) \quad (49)$$

The symplectic matrix $f_{ab}^{(1)}$ represents a $[18 \times 18]$ singular matrix. However, we have commented that there are not more constraints; the noninvertibility of (49) means that the theory has a gauge symmetry. In order to invert the symplectic matrix we choose the following gauge fixing as constraints

$$A_0^i(x) = 0,$$

$$e_0^i(x) = 0,$$

then we introduce the Lagrangians multipliers ϕ_i and α_i associated with the above gauge fixing for constructing a new symplectic Lagrangian. Now the symplectic Lagrangian is given by

$$\mathcal{L}^{(2)} = 2\epsilon^{0ab}\delta_{ij}e_b^i\dot{A}_a^j + \beta\epsilon^{0ab}\delta_{ij}A_b^i\dot{A}_a^j + \Omega\epsilon^{0ab}\delta_{ij}e_b^i\dot{e}_a^j + (\Omega_i^{(0)} + \phi_i)\dot{\lambda}^i + (\beta_i^{(0)} + \alpha_i)\dot{\theta}^i, \quad (50)$$

thus, we identify the following set of symplectic variables $\xi^{(2)a}(x) = \{e_a^i, \lambda^i, \phi_i, A_a^i, \theta^i, \alpha_i\}$ and the symplectic 1-forms $a^{(0)}_a(x) = \{\Omega\epsilon^{0ab}e_{bi}, \Omega_i^{(0)} + \phi_i, 0, 2\epsilon^{0ab}e_{bi} + \beta\epsilon^{0ab}A_{bi}, \beta_i^{(0)} + \alpha_i, 0\}$. Furthermore, by using these symplectic variables we find that the symplectic matrix is given by

$$f_{ab}^{(2)}(x, y) =$$

$$\begin{pmatrix} -2\delta_{ij} & -2(\Omega\delta_{ij}\partial_b + \epsilon_{ijk}(\Omega A_b^k + s | \Lambda | e_b^k)) & 0 & -2\delta_{ij} & -2(\delta_{ij}\partial_b + \epsilon_{ijk}(A_b^k + \Omega e_b^k)) & 0 \\ -2(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(\Omega A_i^k + s | \Lambda | e_a^k)) & 0 & -\delta_i^j & -2(\Omega\delta_{ij}\partial_a - \epsilon_{ijk}(A_i^k + \Omega e_a^k)) & 0 & 0 \\ 0 & \delta_j^i & 0 & 0 & 0 & 0 \\ -2\delta_{ij} & -2(\delta_{ij}\partial_b + \epsilon_{ijk}(A_b^k + \Omega e_b^k)) & 0 & -2\beta\delta_{ij} & -2(\beta\delta_{ij}\partial_b + \epsilon_{ijk}(\beta A_b^k + e_b^k)) & 0 \\ -2(\delta_{ij}\partial_a - \epsilon_{ijk}(A_a^k + \Omega e_a^k)) & 0 & 0 & -2(\beta\delta_{ij}\partial_a - \epsilon_{ijk}(\beta A_a^k + e_a^k)) & 0 & -\delta_i^j \\ 0 & 0 & 0 & 0 & \delta_j^i & 0 \end{pmatrix}$$

$$\times \epsilon^{0ab} \delta^2(x-y) \quad (51)$$

The symplectic matrix $f_{ab}^{(2)}$ represents a $[24 \times 24]$ nonsingular matrix, hence, it is a symplectic tensor. After a long calculation, the inverse is given by

$$[f_{ab}^{(2)}(x, y)]^{-1} = \begin{pmatrix} \frac{\gamma}{2\sqrt{|\Lambda|(s-\gamma^2)}} \epsilon_{0ab} \delta^{ij} & 0 & \delta_j^i \partial_a - \epsilon^i_{jk} A_a^k & -\frac{\gamma^2}{2(s-\gamma^2)} \epsilon_{0ab} \delta^{ij} & 0 & -\epsilon^i_{jk} e_a^k \\ 0 & 0 & \delta_i^j & 0 & 0 & 0 \\ \delta_i^j \partial_b - \epsilon^j_{ik} A_b^k & -\delta_j^i & 0 & -s |\Lambda| \epsilon^j_{ik} e_b^k & 0 & 0 \\ -\frac{\gamma^2}{2(s-\gamma^2)} \epsilon_{0ab} \delta^{ij} & 0 & -s |\Lambda| \epsilon^i_{jk} e_a^k & \frac{s\gamma\sqrt{|\Lambda|}}{2(s-\gamma^2)} \epsilon_{0ab} \delta^{ij} & 0 & \delta_j^i \partial_a - \epsilon^i_{jk} A_a^k \\ 0 & 0 & 0 & 0 & 0 & \delta_i^j \\ -\epsilon^j_{ik} e_b^k & 0 & 0 & \delta_i^j \partial_b - \epsilon^j_{ik} A_b^k & -\delta_j^i & 0 \end{pmatrix} \delta^2(x-y). \quad (52)$$

Therefore, from (52) it is possible to identify the following [FJ] generalized brackets by means of

$$\{\xi_i^{(2)}(x), \xi_j^{(2)}(y)\}_{FD} = [f_{ij}^{(2)}(x, y)]^{-1}, \quad (53)$$

thus, the following brackets are identified

$$\begin{aligned} \{e_a^i(x), e_b^j(y)\}_{FD} &= \frac{\gamma}{2\sqrt{|\Lambda|(s-\gamma^2)}} \epsilon_{0ab} \delta^{ij} \delta^2(x-y), \\ \{A_a^i(x), e_b^j(y)\}_{FD} &= -\frac{\gamma^2}{2(s-\gamma^2)} \epsilon_{0ab} \delta^{ij} \delta^2(x-y), \\ \{A_a^i(x), A_b^j(y)\}_{FD} &= \frac{s\sqrt{|\Lambda|}\gamma}{2(s-\gamma^2)} \epsilon_{0ab} \delta^{ij} \delta^2(x-y), \\ \{e_a^i(x), \phi_j(y)\}_{FD} &= (\delta_j^i \partial_a - \epsilon^i_{jk} A_a^k) \delta^2(x-y), \\ \{A_a^i(x), \alpha_j(y)\}_{FD} &= (\delta_j^i \partial_a - \epsilon^i_{jk} A_a^k) \delta^2(x-y), \\ \{e_a^i(x), \alpha_j(y)\}_{FD} &= -\epsilon^i_{jk} e_a^k, \\ \{A_a^i(x), \phi_j(y)\}_{FD} &= -s |\Lambda| \epsilon^i_{jk} e_a^k, \\ \{\lambda^i(x), \phi_j(y)\}_{FD} &= \delta_i^j \delta^2(x-y), \\ \{\theta^i(x), \alpha_j(y)\}_{FD} &= \delta_i^j \delta^2(x-y). \end{aligned} \quad (54)$$

It is important to comment, that the generalized [FJ] brackets coincide with those obtained by means of the Dirac method reported in the previous section. In fact, if we perform a redefinition of the fields introducing the momenta given by

$$\begin{aligned} \pi_i^a &= s \frac{\sqrt{|\Lambda|}}{\gamma} \epsilon^{0ab} \delta_{ij} e_b^j, \\ \Pi_i^a &= 2\epsilon^{0ab} \delta_{ij} (e_b^j + \frac{1}{2\gamma\sqrt{|\Lambda|}} A_b^j), \end{aligned} \quad (55)$$

the generalized [FJ] brackets (54) take the form

$$\begin{aligned}
\{e^i_a(x), \pi^b_j(y)\}_{FD} &= \frac{s}{2(s-\gamma^2)} \delta^b_a \delta^i_j \delta^2(x-y), \\
\{e^i_a(x), e^j_b(y)\}_{FD} &= \frac{1}{2\sqrt{|\Lambda|}} \frac{\gamma}{s-\gamma^2} \delta^{ij} \epsilon_{0ab} \delta^2(x-y), \\
\{\pi^a_i(x), \pi^b_j(y)\}_{FD} &= \frac{s^2}{2\gamma} \frac{\sqrt{|\Lambda|}}{s-\gamma^2} \delta_{ij} \epsilon^{0ab} \delta^2(x-y), \\
\{A^i_a(x), \Pi^b_j(y)\}_{FD} &= \frac{s-2\gamma^2}{2(s-\gamma^2)} \delta^b_a \delta^i_j \delta^2(x-y), \\
\{A^i_a(x), A^j_b(y)\}_{FD} &= \frac{s\sqrt{|\Lambda|}}{2} \frac{\gamma}{s-\gamma^2} \delta^{ij} \epsilon_{0ab} \delta^2(x-y), \\
\{\Pi^a_i(x), \Pi^b_j(y)\}_{FD} &= \frac{s}{2\gamma\sqrt{|\Lambda|}} \frac{1}{s-\gamma^2} \delta_{ij} \epsilon^{0ab} \delta^2(x-y), \\
\{A^i_a(x), e^j_b(y)\}_{FD} &= \frac{1}{2} \frac{\gamma^2}{\gamma^2-s} \delta^{ij} \epsilon_{0ab} \delta^2(x-y), \\
\{e^i_a(x), \Pi^b_j(y)\}_{FD} &= \frac{1}{2\sqrt{|\Lambda|}} \frac{\gamma}{s-\gamma^2} \delta^b_a \delta^i_j \delta^2(x-y), \\
\{A^i_a(x), \pi^b_j(y)\}_{FD} &= -\frac{s\sqrt{|\Lambda|}}{2} \frac{\gamma}{s-\gamma^2} \delta^b_a \delta^i_j \delta^2(x-y), \\
\{\pi^a_i(x), \Pi^b_j(y)\}_{FD} &= \frac{1}{2} \frac{s}{s-\gamma^2} \delta_{ij} \epsilon^{0ab} \delta^2(x-y),
\end{aligned} \tag{56}$$

where we can observe that coincide with the full Dirac's brackets found in (27).

Furthermore, we have commented above that in [FJ] approach it is not necessary classify the constraints in first class and second class, in [FJ] formulation all the constraints are at the same footing. Thus, we can carry out the counting of physical degrees of freedom in the following form; there are 12 dynamical variables (e^i_a, A^i_a) and 12 constraints ($\Omega_i^{(0)}, \beta_i^{(0)}, A^i_0, e^i_0$), therefore, the theory lacks of physical degrees of freedom.

We finish this section by calculating the gauge transformations of the theory, for this aim we calculate the modes of the matrix (49)

$$(w^{(1)})_1^T = (-\delta^i_j \partial_a \delta^2(x-y) - \epsilon^i_{jk} A^k_a \delta^2(x-y), \delta^i_j \delta^2(x-y), -s | \Lambda | \epsilon^i_{jk} e^k_a \delta^2(x-y), 0), \tag{57}$$

$$(w^{(1)})_2^T = (-\epsilon^i_{jk} e^k_a \delta^2(x-y), 0, -\delta^i_j \partial_a \delta^2(x-y) - \epsilon^i_{jk} A^k_a \delta^2(x-y), \delta^i_j \delta^2(x-y)). \tag{58}$$

In agreement with the [FJ] symplectic formalism, the zero-modes $(w^{(1)})_1^T$ and $(w^{(1)})_2^T$ are the generators of infinitesimal gauge transformation of the action (35) and are given by

$$\begin{aligned}
\delta e^i_a(x) &= -D_a \varepsilon^i + \epsilon^i_{jk} e^k_a \tau^j, \\
\delta e^i_0(x) &= \partial_0 \varepsilon^i, \\
\delta A^i_a(x) &= -D_a \tau^i + s | \Lambda | \epsilon^i_{jk} e^k_a \varepsilon^j, \\
\delta A^i_0(x) &= \partial_0 \tau^i.
\end{aligned}$$

In fact, the mode (57) is the generator of translations and the mode (58) is the generator of rotations. In this manner, by using the [FJ] symplectic framework we have reproduced the Λ -deformed $ISO(3)$

gauge transformations reported by means of Dirac's method. Finally, in order to complete our work, in Appendix A we have developed the Dirac analysis for the Abelian case. In that appendix, we performed the full constraints program and we have constructed the Dirac brackets by fixing the gauge, then in Appendix B we reproduce all Dirac's results in a more economical way by using [FJ] framework.

V. CONCLUSIONS AND PROSPECTS

In this paper a pure Dirac's formalism and a full [FJ] approach for [BL] model have been performed. With respect to Dirac's method, the complete structure of the constraints was found and the algebra between the constraints defined on the full phase space was developed. Furthermore, we observed that the internal group is $SU(2)$ (or $SO(3)$), however, the fundamental gauge symmetry correspond to an Λ -deformed $ISO(3)$ transformations. In addition, we have eliminated the second class constraints by introducing the Dirac brackets, and we observed that the Dirac brackets depends on the γ parameter and this fact makes classically different [BL] from Palatini's theory. On the other hand, we have reproduced all the relevant Dirac's results by performing the [FJ] framework. In fact, we have found the [FJ] constraints and we have showed that there are less constraints than those found with Dirac's method. Moreover, we have showed that the generalized [FJ] brackets and the Dirac's ones coincide to each other. In this manner, we have reproduced all relevant Dirac's results by working with [FJ], in particular we can see that [FJ] is more economical than Dirac's method. In addition, we proved the equivalence between [BL] model and Chern-Simons theory; from a pure Dirac's analysis of the Chern-Simons theory, all relevant Dirac's results obtained using the connection and the frame fields as dynamical variables were reproduced. Finally, we would to comment that our [FJ] analysis is generic and we can reproduce all the results reported in [4] where Dirac's method was employed. In fact, it is straightforward observe that using the Axial gauge in the matrix (49), the [BL] theory can be written in terms of a $SO(3)$ Ashtekar connection. This result is obtained in more economical way by using the [FJ] framework. Hence, our results extend those results found in the literature.

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Appendix A: Canonical analysis of the [BL] Abelian theory

In this appendix, we shall resume the canonical analysis of the Abelian version of [BL] action given by

$$\begin{aligned}
S^{Abelian}[A, e] &= \int 2e^i \wedge F_i[A] + \frac{1}{\sqrt{|\Lambda|}} \int A^i \wedge dA_i + s\sqrt{|\Lambda|} e^i \wedge de_i \\
&= \int \epsilon^{0ab} \left[\left(\frac{A_0^i}{\gamma\sqrt{|\Lambda|}} + e_0^i \right) F_{abi} + \left(\frac{A_b^i}{\gamma\sqrt{|\Lambda|}} + e_b^i \right) \dot{A}_{ai} + \left(\frac{s\sqrt{|\Lambda|}}{\gamma} e_0^i + A_0^i \right) T_{abi} \right. \\
&\quad \left. + \left(\frac{s\sqrt{|\Lambda|}}{\gamma} e_b^i + A_b^i \right) \dot{e}_{ai} \right], \tag{A1}
\end{aligned}$$

where A_μ^a and e_μ^a are a set of three $U(1)$ vector fields, $F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i$, $T_{ab} = \partial_a e_b^i - \partial_b e_a^i$. By introducing the canonical momenta defined by

$$\pi_i^\lambda := \frac{\partial \mathcal{L}}{\partial \dot{A}_\lambda^i} = \epsilon^{0\lambda\rho} \left[\frac{1}{\sqrt{|\Lambda|}\gamma} A_{\rho i} + e_{\rho i} \right], \tag{A2}$$

$$p_i^\lambda := \frac{\partial \mathcal{L}}{\partial \dot{e}_\lambda^i} = \epsilon^{0\lambda\rho} \left[A_{\rho i} + \frac{s\sqrt{|\Lambda|}}{\gamma} e_{\rho i} \right]. \tag{A3}$$

and performing the canonical analysis, we obtain the following results: we find 4 first class constraints

$$\begin{aligned}
\gamma^1 &= p_i^0 \approx 0, \\
\gamma^2 &= 2\partial_a p_i^a - \partial_a \phi_i^a \approx 0, \\
\gamma^3 &= \pi_i^0 \approx 0, \\
\gamma^4 &= 2\partial_a \pi_i^a - \partial_a \Phi_i^a \approx 0, \tag{A4}
\end{aligned}$$

and the following 4 second class constraints

$$\begin{aligned}
\chi_{1i}^a &= p_i^a - \epsilon^{0ab} [A_{bi} + \Omega e_{bi}] \approx 0, \\
\chi_{2i}^a &= \pi_i^a - \epsilon^{0ab} [\beta A_{bi} + e_{bi}] \approx 0. \tag{A5}
\end{aligned}$$

Now, the nontrivial algebra between the constraints is given by the algebra of the second class constraints as expected

$$\begin{aligned}
\{\chi_{1i}^a(x), \chi_{1j}^b(y)\} &= -2\Omega \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\
\{\chi_{1i}^a(x), \chi_{2j}^b(y)\} &= -2\epsilon^{0ab} \delta_{ij} \delta^2(x-y), \\
\{\chi_{2i}^a(x), \chi_{2j}^b(y)\} &= -2\beta \epsilon^{0ab} \delta_{ij} \delta^2(x-y). \tag{A6}
\end{aligned}$$

In order to construct the Dirac brackets by eliminating the second class constraints, we write in matrix form the Poisson brackets among second class constraints, namely

$$C^{ab} = \begin{pmatrix} -2\Omega & -2 \\ -2 & -2\beta \end{pmatrix} \epsilon^{0ab} \delta_{ij} \delta^2(x-y), \tag{A7}$$

and we calculate its inverse given by

$$[C^{ab}]^{-1} = \frac{\gamma^2}{2(s - \gamma^2)} \begin{pmatrix} \beta & -1 \\ -1 & \Omega \end{pmatrix} \epsilon_{0ab} \delta^{ij} \delta^2(x - y). \quad (\text{A8})$$

Hence, by using the matrix (A8) we obtain the following Dirac's brackets of the theory

$$\begin{aligned} \{e_a^i(x), e_b^j(y)\}_D &= \frac{\gamma}{2\sqrt{|\Lambda|}(s - \gamma^2)} \epsilon_{0ab} \delta^{ij} \delta^2(x - y), \\ \{A_a^i(x), e_b^j(y)\}_D &= -\frac{\gamma^2}{2(s - \gamma^2)} \epsilon_{0ab} \delta^{ij} \delta^2(x - y), \\ \{e_a^i(x), p_j^b(y)\}_D &= \frac{1}{2} \delta_a^b \delta_j^i \delta^2(x - y), \\ \{p_i^a(x), p_j^b(y)\}_D &= \frac{s\sqrt{|\Lambda|}}{2\gamma} \epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{A_a^i(x), A_b^j(y)\}_D &= \frac{s\sqrt{|\Lambda|}\gamma}{2(s - \gamma^2)} \epsilon_{0ab} \delta^{ij} \delta^2(x - y), \\ \{A_a^i(x), \pi_j^b(y)\}_D &= \frac{1}{2} \delta_a^b \delta_j^i \delta^2(x - y), \\ \{\pi_i^a(x), \pi_j^b(y)\}_D &= \frac{\beta}{2} \epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{p_i^a(x), \pi_j^b(y)\}_D &= \frac{1}{2} \epsilon^{0ab} \delta_{ij} \delta^2(x - y), \\ \{e_0^i(x), p_j^0(y)\}_D &= \delta_j^i \delta^2(x - y), \\ \{A_0^i(x), \pi_j^0(y)\}_D &= \delta_j^i \delta^2(x - y), \end{aligned} \quad (\text{A9})$$

hence, the Dirac brackets for Abelian and non-Abelian theory coincide to each other. In the following lines, we will construct the Dirac brackets by fixing the gauge, then we will reproduce these results by means [FJ] framework.

1. Dirac's brackets by fixing the gauge

In order to construct the Dirac brackets by fixing the gauge, it is necessary to convert the first class constraints into second class, we will work with the temporal and Coulomb gauge

$$\begin{aligned} \Omega_1 &= e_0^i \approx 0, \\ \Omega_2 &= \partial^a e_a^i \approx 0, \\ \Omega_3 &= A_0^i \approx 0, \\ \Omega_4 &= \partial^a A_a^i \approx 0, \\ \Omega_5 &= p_i^0 \approx 0, \\ \Omega_6 &= 2\partial_a p_i^a - \partial_a \chi_{1i}^a \approx 0, \\ \Omega_7 &= \pi_i^0 \approx 0, \\ \Omega_8 &= 2\partial_a \pi_i^a - \partial_a \chi_{2i}^a \approx 0, \\ \Omega_9 &= p_i^a - \epsilon^{0ab} [A_{bi} + \Omega e_{bi}] \approx 0, \\ \Omega_{10} &= \pi_i^a - \epsilon^{0ab} [\beta A_{bi} + e_{bi}] \approx 0. \end{aligned} \quad (\text{A10})$$

in this manner, the matrix whose entries are the Poisson brackets between the constraints, namely G , is given by

$$G(x, y) = \begin{pmatrix} -2\Omega\epsilon^{0ab}\delta_{ij} & 0 & 0 & 0 & \delta_i^j\partial_x^a & -2\epsilon^{0ab}\delta_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_j^i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_i^j\nabla_x^2 & 0 & 0 & 0 & 0 & 0 \\ \delta_j^i\partial_x^b & 0 & 0 & -\delta_j^i\nabla_x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\epsilon^{0ab}\delta_{ij} & 0 & 0 & 0 & 0 & -2\beta\epsilon^{0ab}\delta_{ij} & 0 & 0 & 0 & \delta_i^j\partial_x^a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_i^j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_j^i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_i^j\nabla_x^2 \\ 0 & 0 & 0 & 0 & 0 & \delta_j^i\partial_x^b & 0 & 0 & -\delta_j^i\nabla_x^2 & 0 \end{pmatrix} \times \delta^2(x - y). \quad (\text{A11})$$

Hence, the inverse of G becomes

$$[G(x, y)]^{-1} = \begin{pmatrix} \epsilon_{0ab}\delta^{ij}\frac{\beta}{2\theta} & 0 & 0 & \epsilon_{0ab}\delta_j^i\frac{\beta\partial_x^b}{2\theta\nabla_x^2} & 0 & -\epsilon_{0ab}\delta^{ij}\frac{1}{2\theta} & 0 & 0 & -\epsilon_{0ab}\delta_j^i\frac{\partial_x^b}{2\theta\nabla_x^2} & 0 \\ 0 & 0 & \delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\delta_j^i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon_{0ba}\delta_j^i\frac{\beta\partial_x^a}{2\theta\nabla_x^2} & 0 & 0 & 0 & -\frac{\delta_j^i}{\nabla_x^2} & -\epsilon_{0ba}\delta_j^i\frac{\partial_x^a}{2\theta\nabla_x^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta_j^i}{\nabla_x^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\epsilon_{0ab}\delta^{ij}\frac{1}{2\theta} & 0 & 0 & -\epsilon_{0ab}\delta_j^i\frac{\partial_x^b}{2\theta\nabla_x^2} & 0 & \epsilon_{0ab}\delta^{ij}\frac{\Omega}{2\theta} & 0 & 0 & \epsilon_{0ab}\delta_j^i\frac{\Omega\partial_x^b}{2\theta\nabla_x^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_j^i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta_j^i & 0 & 0 & 0 \\ -\epsilon_{0ba}\delta_j^i\frac{\partial_x^a}{2\theta\nabla_x^2} & 0 & 0 & 0 & 0 & \epsilon_{0ba}\delta_j^i\frac{\Omega\partial_x^a}{2\theta\nabla_x^2} & 0 & 0 & 0 & -\frac{\delta_j^i}{\nabla_x^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\delta_j^i}{\nabla_x^2} & 0 \end{pmatrix} \times \delta^2(x - y),$$

here we have defined $\theta = \beta\Omega - 1$. Finally, we use the inverse matrix G^{-1} and we find the following Dirac's brackets

$$\begin{aligned} \{e_a^i(x), p_j^b(y)\}_D &= \delta_j^i(\delta_a^b - \frac{\partial_a\partial^b}{\nabla^2})\delta(x - y), \\ \{e_a^i(x), e_b^j(y)\}_D &= 0, \\ \{p_i^a(x), p_j^b(y)\}_D &= 0, \\ \{A_a^i(x), \pi_j^b(y)\}_D &= \delta_j^i(\delta_a^b - \frac{\partial_a\partial^b}{\nabla^2})\delta(x - y), \\ \{A_a^i(x), A_b^j(y)\}_D &= 0, \\ \{\pi_i^a(x), \pi_j^b(y)\}_D &= 0. \end{aligned} \quad (\text{A12})$$

In the following section, we will reproduce these results by mean of [FJ] formalism.

Appendix B: Faddeev-Jackiw analysis of [BL] Abelian theory by working with the phase space

Now, in this section we shall study the action (A1) by means [FJ] formalism, we shall work with the phase space as symplectic variables [15]. Hence, the Lagrangian density can be written as

$$\mathcal{L} = \epsilon^{0ab} \left[\left(\frac{A_b^i}{\gamma\sqrt{|\Lambda|}} + e_b^i \right) \dot{A}_{ai} + \left(\frac{s\sqrt{|\Lambda|}}{\gamma} e_b^i + A_b^i \right) \dot{e}_{ai} \right] - V^{(0)}, \quad (\text{B1})$$

where the potential symplectic is given by

$$V^{(0)} = -\epsilon^{0ab} \left[\left(\frac{A_0^i}{\gamma \sqrt{|\Lambda|}} + e_0^i \right) F_{abi} + \left(\frac{s \sqrt{|\Lambda|}}{\gamma} e_0^i + A_0^i \right) T_{abi} \right]. \quad (\text{B2})$$

By introducing the canonical momenta

$$\begin{aligned} p_i^a &= \epsilon^{0ab} (A_{bi} + \Omega e_{bi}), \\ \pi_i^a &= \epsilon^{0ab} (e_{bi} + \beta A_{bi}), \end{aligned} \quad (\text{B3})$$

and writing the fields in the following form

$$\begin{aligned} \epsilon^{0ab} e_{bi} &= \frac{s}{s - \gamma^2} (\beta p_j^a - \pi_j^a), \\ \epsilon^{0ab} A_{bi} &= \frac{s}{s - \gamma^2} (\Omega \pi_j^a - p_j^a), \end{aligned}$$

the first-order symplectic Lagrangian density takes the form

$$\mathcal{L}^{(0)} = \pi_i^a \dot{A}_a^i + p_i^a \dot{e}_a^i - V^{(0)}, \quad (\text{B4})$$

where the potential symplectic $V^{(0)}$ is given by

$$V^{(0)} = -2A_0^i \partial_a \pi_i^a - 2e_0^i \partial_a p_i^a. \quad (\text{B5})$$

In this manner, we can identify the corresponding symplectic variables $\xi^{(0)a}(x) = \{e_a^i, p_i^a, e_0^i, A_a^i, \pi_i^a, A_0^i\}$ and the symplectic 1-form $a^{(0)}_a(x) = \{p_i^a, 0, 0, \pi_i^a, 0, 0\}$, thus, by using the symplectic variables, the symplectic matrix takes the form

$$f_{ab}^{(0)}(x, y) = \begin{pmatrix} 0 & -\delta^a_b \delta^j_i & 0 & 0 & 0 & 0 \\ \delta^b_a \delta^i_j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^a_b \delta^j_i & 0 \\ 0 & 0 & 0 & \delta^a_b \delta^i_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x - y), \quad (\text{B6})$$

This matrix is singular, this means that the theory has constraints. The zero modes of this matrix are given by $(v_a^{(0)})_1^T = (0, 0, v^{e_0^i}, 0, 0, 0)$ and $(v_a^{(0)})_2^T = (0, 0, 0, 0, 0, v^{A_0^i})$, where $v^{e_0^i}$ and $v^{A_0^i}$ are arbitrary functions. Now, by using the zero-modes we can get the following constraints

$$\begin{aligned} 0 &= \int d^2x (v^{(0)})_a^T(x) \frac{\delta}{\delta \xi^{(0)i}(x)} \int d^2y V^{(0)}(\xi) \\ &= \int d^2x v^{e_0^i}(x) [-2\partial_a p_i^a] \\ &\rightarrow \Omega_i^{(0)} := -2\partial_a p_i^a = 0, \end{aligned} \quad (\text{B7})$$

$$\begin{aligned}
0 &= \int d^2x (v^{(0)})_a^T(x) \frac{\delta}{\delta \xi^{(0)i}(x)} \int d^2y V^{(0)}(\xi) \\
&= \int d^2x v^{A_0}(x) [-2\partial_a \pi_i^a] \\
&\rightarrow \Theta_i^{(0)} := -2\partial_a \pi_i^a = 0.
\end{aligned} \tag{B8}$$

On the other hand, we will research if there are more constraints by calculating the following contraction [16]

$$f_{cd}^{(1)} \xi^d = Z_c(\xi), \tag{B9}$$

where

$$f_{cd}^{(1)} = \left(\begin{array}{c} f_{ab}^{(0)} \\ \frac{\partial \Omega^{(0)}}{\partial \xi^a} \end{array} \right) = \begin{pmatrix} 0 & -\delta^a_b \delta^j_i & 0 & 0 & 0 & 0 \\ \delta^b_a \delta^i_j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^a_b \delta^j_i & 0 \\ 0 & 0 & 0 & \delta^b_a \delta^j_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\partial_b \delta^j_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\partial_b \delta^j_i & 0 \end{pmatrix} \delta^2(x-y), \tag{B10}$$

and

$$Z_c(\xi) = \begin{pmatrix} \frac{\partial V^{(0)}(\xi)}{\partial \xi^a} \\ 0 \end{pmatrix}, \tag{B11}$$

The matrix $(f_{ab}^{(1)})$ given in (B10) is not obviously a square matrix, but it still has linearly independent modes given by $(v^{(1)})_1^T = (2\partial_a v^\lambda, 0, v^{e_0}, 0, 0, 0, v^\lambda, 0)$ and $(v^{(1)})_2^T = (0, 0, 0, 2\partial_a v^\alpha, 0, v^{A_0}, 0, v^\alpha)$. Multiplication of $(f_{cd}^{(1)})$ by $(v^{(1)})_c^T$ from the left side gives zero. The contraction of these modes reads

$$(v^{(1)})_c^T Z_c \big|_{\Omega^{(0)}=0} = 0, \tag{B12}$$

which is an identity, hence, there is not new constraints.

Furthermore, we use the following Lagrangian multipliers (λ^i, ρ^i) enforcing the constraints (B7) and (B8) in order to construct a new symplectic Lagrangian

$$\mathcal{L}^{(1)} = \pi_i^a \dot{A}_a^i + p_i^a \dot{e}_a^i + (2\partial_a p_i^a) \dot{\lambda}^i + (2\partial_a \pi_i^a) \dot{\rho}^i - V^{(1)}, \tag{B13}$$

where $V^{(1)} = V^{(0)} \big|_{\partial_a \pi_i^a=0, \partial_a p_i^a=0} = 0$, is the symplectic potential. From (B13) we identify the following symplectic variables $\xi^{(1)a}(x) = \{e_a^i, p_i^a, \lambda^i, A_a^i, \pi_i^a, \rho^i\}$, and the 1-forms $a^{(1)}_a(x) =$

$\{p_i^a, 0, 2\partial_a p_i^a, \pi_i^a, 0, 2\partial_a \pi_i^a\}$, thus, the corresponding symplectic matrix is given by

$$f_{ab}^{(1)}(x, y) = \begin{pmatrix} 0 & -\delta^a_b \delta^j_i & 0 & 0 & 0 & 0 \\ \delta^b_a \delta^i_j & 0 & -2\delta^i_j \partial_a & 0 & 0 & 0 \\ 0 & -2\delta^j_i \partial_b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^a_b \delta^j_i & 0 \\ 0 & 0 & 0 & \delta^b_a \delta^i_j & 0 & -2\delta^i_j \partial_a \\ 0 & 0 & 0 & 0 & -2\delta^j_i \partial_b & 0 \end{pmatrix} \delta^2(x - y), \quad (\text{B14})$$

the matrix is still singular, but we have proved, however, that there are not new constraints. Therefore this system has a gauge symmetry. In order to obtain a symplectic tensor, we will fix the gauge, let us fix the Coulomb gauge $\partial^a e_a^i = 0$, $\partial^a A_a^i = 0$ and we will introduce this information by constructing a new symplectic Lagrangian adding new Lagrange multiples, namely ϕ_i and θ_i , enforcing the gauge fixing, we obtain

$$\mathcal{L}^{(2)} = \pi_i^a \dot{A}_a^i + p_i^a \dot{e}_a^i + (2\partial_a p_i^a) \dot{\lambda}^i + (2\partial_a \pi_i^a) \dot{\rho}^i + (\partial^a e_a^i) \dot{\phi}_i + (\partial^a A_a^i) \dot{\theta}_i, \quad (\text{B15})$$

now the symplectic variables are given by $\xi^{(2)}(x) = \{e_a^i, p_i^a, \lambda^i, \phi_i, A_a^i, \pi_i^a, \rho^i, \theta_i\}$ and the 1-forms $a^{(2)}(x) = \{p_i^a, 0, 2\partial_a p_i^a, \partial^a e_a^i, \pi_i^a, 0, 2\partial_a \pi_i^a, \partial^a A_a^i\}$. In this manner, the symplectic matrix takes the form

$$f_{ab}^{(2)}(x, y) = \begin{pmatrix} 0 & -\delta^a_b \delta^j_i & 0 & -\delta^j_i \partial_a & 0 & 0 & 0 & 0 \\ \delta^b_a \delta^i_j & 0 & -2\delta^i_j \partial_a & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\delta^j_i \partial_b & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta^i_j \partial_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta^a_b \delta^j_i & 0 & -\delta^j_i \partial_a \\ 0 & 0 & 0 & 0 & \delta^b_a \delta^i_j & 0 & -2\delta^i_j \partial_a & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\delta^j_i \partial_b & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^i_j \partial_b & 0 & 0 & 0 \end{pmatrix} \times \delta^2(x - y), \quad (\text{B16})$$

where we can observe that $f_{ab}^{(2)}(x, y)$ is an symplectic tensor and therefore is invertible. The inverse matrix of $f_{ab}^{(2)}(x, y)$ is given by

$$[f_{ab}^{(2)}(x, y)]^{-1} =$$

$$\begin{pmatrix}
0 & \delta^i_j(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2}) & 0 & -\delta^i_j \frac{\partial_a}{\nabla^2} & 0 & 0 & 0 & 0 \\
-\delta^j_i(\delta^a_b - \frac{\partial_b \partial^a}{\nabla^2}) & 0 & -\frac{1}{2}\delta^j_i \frac{\partial^a}{\nabla^2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2}\delta^i_j \frac{\partial^b}{\nabla^2} & 0 & -\delta^i_j \frac{1}{2} \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 \\
-\delta^j_i \frac{\partial_b}{\nabla^2} & 0 & \frac{1}{2}\delta^j_i \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta^i_j(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2}) & 0 & -\delta^i_j \frac{\partial_a}{\nabla^2} \\
0 & 0 & 0 & 0 & -\delta^j_i(\delta^a_b - \frac{\partial_b \partial^a}{\nabla^2}) & 0 & -\frac{1}{2}\delta^j_i \frac{\partial^a}{\nabla^2} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\delta^i_j \frac{\partial^b}{\nabla^2} & 0 & -\frac{1}{2}\delta^i_j \frac{1}{\nabla^2} \\
0 & 0 & 0 & 0 & -\delta^j_i \frac{\partial_b}{\nabla^2} & 0 & \frac{1}{2}\delta^j_i \frac{1}{\nabla^2} & 0
\end{pmatrix}
\times \delta^2(x-y) \tag{B17}$$

where it is possible identify the [FJ] generalized brackets as

$$\{\xi_a^{(2)}(x), \xi_b^{(2)}(y)\}_{FD} = [f_{ab}^{(2)}(x, y)]^{-1}, \tag{B18}$$

thus, we find the following brackets

$$\begin{aligned}
\{e_a^i(x), p_j^b(y)\}_{FD} &= [f_{12}^{(2)}(x, y)]^{-1} = \delta^i_j(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2})\delta^2(x-y), \\
\{A_a^i(x), \pi_j^b(y)\}_{FD} &= [f_{56}^{(2)}(x, y)]^{-1} = \delta^i_j(\delta^b_a - \frac{\partial_a \partial^b}{\nabla^2})\delta^2(x-y), \\
\{e_a^i(x), e_b^j(y)\}_{FD} &= [f_{11}^{(2)}(x, y)]^{-1} = 0, \\
\{A_a^i(x), A_b^j(y)\}_{FD} &= [f_{55}^{(2)}(x, y)]^{-1} = 0, \\
\{p_i^a(x), p_j^b(y)\}_{FD} &= [f_{22}^{(2)}(x, y)]^{-1} = 0, \\
\{\pi_i^a(x), \pi_j^b(y)\}_{FD} &= [f_{66}^{(2)}(x, y)]^{-1} = 0, \\
\{p_i^a(x), \lambda^j(y)\}_{FD} &= [f_{23}^{(2)}(x, y)]^{-1} = -\frac{1}{2}\delta_i^j \frac{\partial^a}{\nabla^2} \delta^2(x-y), \\
\{\pi_i^a(x), \rho^j(y)\}_{FD} &= [f_{67}^{(2)}(x, y)]^{-1} = -\frac{1}{2}\delta_i^j \frac{\partial^a}{\nabla^2} \delta^2(x-y), \\
\{\lambda^i(x), \phi_j(y)\}_{FD} &= [f_{34}^{(2)}(x, y)]^{-1} = -\frac{1}{2}\delta_j^i \frac{1}{\nabla^2} \delta^2(x-y), \\
\{\rho^i(x), \theta_j(y)\}_{FD} &= [f_{78}^{(2)}(x, y)]^{-1} = -\frac{1}{2}\delta_j^i \frac{1}{\nabla^2} \delta^2(x-y), \\
\{e_a^i(x), \phi_j(y)\}_{FD} &= [f_{14}^{(2)}(x, y)]^{-1} = -\delta_j^i \frac{\partial_a}{\nabla^2} \delta^2(x-y), \\
\{A_a^i(x), \theta_j(y)\}_{FD} &= [f_{58}^{(2)}(x, y)]^{-1} = -\delta_j^i \frac{\partial_a}{\nabla^2} \delta^2(x-y).
\end{aligned} \tag{B19}$$

We can observe that the Dirac brackets given in (A12) and the [FJ] generalized brackets given in (B19) coincide to each other.

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